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A REVIEW OF METHODS FOR PREDICTING HEAT-TRANSFER COEFFICIENTS FOR LAMINAR UNIFORM-PROPERTY BOUNDARY LAYER FLOWS

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Abstract—Fifteen methods for predicting laminar heat transfer coefficients are analysed and classified. Then each is applied to the problem of calculating the distribution of Nusselt number around the leading half of a circular cylinder in laminar flow. Large differences are shown to exist beween the predictions of the theories.

Comparison is made with the "exact" solution of Frössling and the experimental data of Schmidt and Wenner. Short of more reliable experimental data, no final conclusions can be drawn about the relative accuracy of the theoretical methods.

NOTATION

c, specific heat of fluid, (equation P-3), (Btu/lb°F);

E, a function used in Skopets' method, (Table 1);

f, dimensionless stream function defined by $f \equiv (\psi/u_G)\sqrt{[(\mathrm{d}u_G/\mathrm{d}x)(1/\beta\nu)]}$, ($\psi$ being defined by $u \equiv \partial\psi/\partial y$), (equation OS-1), (—);

G, $\equiv \sigma^{-1/3}[(1/\beta)(\Delta_4^2/\nu)(\mathrm{d}u_G/\mathrm{d}x)]^{-1/2}$, (equation OI-3), (—);

H_{12}, ratio of displacement thickness to momentum thickness, (equation A-1), (—);

H_{24}, ratio of momentum thickness to shear thickness, (equation A-2), (—);

h, local heat transfer coefficient $\equiv -k(\partial t/\partial y)_s/(t_s - t_G)$, (Btu/ft²h °F);

k, thermal conductivity of fluid, (equation P-3), (Btu/ft h°F);

L, reference length of the body; diameter for circular cylinder, (ft);

n, $\equiv u_G(\mathrm{d}^2u_G/\mathrm{d}x^2)/(\mathrm{d}u_G/\mathrm{d}x)^2$, (—);

Nu, local Nusselt number, $\equiv hL/k$, (—);

p, $\equiv [(1/\beta)(\Delta_4\Delta_2/\nu)(\mathrm{d}u_G/\mathrm{d}x)]^{1/3}$; thus for "similar" flows $p = \sigma^{-1/3}$, from equations (43) and (46) in [26], (equation OI-3), (—);

Re, Reynolds number, $\equiv UL/\nu$, (—);

t, temperature of fluid, (equation P-3), (°F);

u, flow velocity in x-direction, (equation P-1), (ft/h);

U, reference flow velocity, (ft/h);

v, flow velocity in y-direction, (equation P-1), (ft/h);

x, distance measured along wall in same direction as mainstream (ft);

y, distance measured normal to the wall, (ft);

β, parameter relating to acceleration of mainstream, $\beta \equiv 1/(1 - n/2)$, (equation OS-1);

δ, "A" thickness of the velocity boundary layer, (ft);

δ_1, displacement thickness, $\equiv \int_0^\infty (1 - u/u_G)\,\mathrm{d}y$, (ft);

δ_2, momentum thickness, $\equiv \int_0^\infty (u/u_G)(1 - u/u_G)\,\mathrm{d}y$, (ft);

δ_4, shear thickness, $\equiv u_G/(\partial u/\partial y)_s$;

$\delta_{D.P.}$ defined as the distance from the surface

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to the point where the dynamic pressure is one-half of its value outside the boundary layer, (Table 1), (ft);

Δ, "A" thickness of the thermal boundary layer, (ft);

Δ_1, thermal displacement thickness, $\equiv \int_0^\infty [(t - t_G)/(t_s - t_G)]\, dy$, (ft);

Δ_2, convection thickness, $\equiv \int_0^\infty (u/u_G)\, [(t - t_G)/(t_s - t_G)]\, dy$, (ft);

Δ_4, conduction thickness, $\equiv -(t_s - t_G)/(\partial t/\partial y)_s$, (ft);

Δ_m, "mixed thermal thickness", $\equiv \sqrt{[\Delta_4^2 + 11{\cdot}82\Delta_2^2]}$;

η, Dimensionless space co-ordinate, $\equiv y\ \sqrt{[(1/\nu\ \beta)\ (du_G/dx)]}$, (equation OS-1), (—);

θ, Dimensionless temperature difference, $\equiv (t - t_s)/(t_G - t_s)$, (equation OS-2), (—);

μ, dynamic viscosity of fluid, (lb/ft h);

ν, kinematic viscosity of fluid, (ft^2/h);

ρ, fluid density, (lb/ft^3);

σ, Prandtl number, $\equiv c\mu/k$, (equation OS-2), (—);

τ, shear stress in fluid, (lbf/ft^2).

Subscripts

G, stream conditions just outside boundary layer;

s, fluid conditions adjacent to the surface.

1. INTRODUCTION

The problem considered

DURING the last twenty years, numerous approximate methods have been proposed for calculating heat-transfer coefficients at points on the surface of an isothermal body. When applied to a particular problem, the amounts of computation involved vary greatly from one method to another. However, no comprehensive survey of the relative accuracy of the methods has been made; there is therefore no way of establishing whether accuracy increases with computational labour, or of deciding which method is preferable for a given purpose.

The present paper does not fully answer the questions just raised, but it does throw some light on them: each of the methods is applied to the same problem, that of heat transfer from a cylinder in cross-flow, and the results are compared. Experimental data of Schmidt and Wenner [1] are available for this situation, and the theoretical predictions can be compared with these. Unfortunately, as has been demonstrated by Kestin *et al.* [2], the presence of even small intensities of turbulence in the free stream can appreciably augment the heat transfer coefficient; consequently, there are still no experimental data available for situations which are sufficiently close to that postulated in the calculation to act as standards of comparison for the theoretical predictions.

Mode of presentation

It would be impractical to describe each of the theoretical methods in details. Instead we attempt to display their essential features by means of Table 1. In addition to compactness, this device has the advantage of permitting differences and similarities between methods to be easily discerned; it further permits the description of the methods in the text to be kept in general terms

Since this description necessarily relates to mathematical equations and concepts, these are introduced in section 2. Derivations are not given however, since these may be found in recent papers published in the present journal [3, 4, 5]. Section 3 provides the discussion of Table 1 and of calculation methods in general. The results of applying the methods to the circular cylinder problem will be found in section 4.

2. EQUATIONS

2.1. *Partial differential equations*

For a fluid of uniform properties flowing in a two-dimensional,* steady, laminar boundary layer, at low Mach Numbers, the exact calculation of the heat transfer rate requires the solution of the following partial differential equations:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(P-1)}$$

* The existence of the Mangler Transformation [6] connecting two-dimensional and axi-symmetrical flows permits us to restrict attention to the former.

Momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_G\frac{du_G}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(P-2)}$$

Energy:

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\frac{k}{c\rho}\frac{\partial^2 t}{\partial y^2}. \qquad \text{(P-3)}$$

The boundary conditions which we shall here consider are those corresponding to an isothermal impervious wall, i.e.: $u=v=0$ and $t=t_S$ at $y=0$; and $u=u_G$ and $t=t_G$ at $y=\infty$.

2.2. *Ordinary differential equations for "similar" boundary layers*

For flows in which du_G/dx is proportional to $u_G{}^n$, where n is a constant, it is shown in [3] and [5] that the above partial differential equations reduce to the following ordinary ones:

$$f'''+ff''+\beta(1-f'^2)=0 \qquad \text{(OS-1)}$$

and

$$\theta''+\sigma f\theta'=0 \qquad \text{(OS-2)}$$

where the prime signifies differentiation with respect to the dimensionless distance, η.

Here the boundary conditions are:

$$f=f'=\theta=0 \text{ at } \eta=0;\ f'=\theta=1 \text{ at } \eta\to\infty.$$

2.3. *Ordinary differential equations obtained by integration*

The partial differential equations (P-1), (P-2) and (P-3) may be integrated formally with respect to y. There results:

Integral momentum equation:

$$\frac{u_G}{\nu}\frac{d\delta_2^2}{dx}=2\frac{\delta_2}{\delta_4}-2\left(2+\frac{\delta_1}{\delta_2}\right)\frac{\delta_2^2}{\nu}\frac{du_G}{dx} \qquad \text{(OI-1)}$$

Integral energy equation:

$$\frac{u_G}{\nu}\frac{d\Delta_2^2}{dx}=\frac{2}{\sigma}\frac{\Delta_2}{\Delta_4}-2\frac{\Delta_2^2}{\nu}\frac{du_G}{dx}. \qquad \text{(OI-2)}$$

The integral energy equation may also be written in the form

$$\frac{d}{dx}\left[u_G\left(\frac{1}{\beta}\frac{du_G}{dx}\right)^{-1/2}p^2G\right]=\frac{1}{\sigma}\left(\frac{1}{\beta}\frac{du_G}{dx}\right)^{1/2}\frac{G}{p}. \qquad \text{(OI-3)}$$

Equations (P-2) and (P-3) may also be multiplied by y^k, where $k=1, 2, 3, \ldots$, before integration with respect to y, which thereupon gives the "kth moment" integral momentum equation and the "kth moment" integral energy equation respectively.

2.4. *Ordinary differential equations based on guesswork*

In "similar" boundary layers, the quantities δ_2/δ_4 and δ_1/δ_2 depend only on (δ_2^2/ν) (du_G/dx), while the quantities Δ_2/Δ_4 depend only on (Δ_2^2/ν) (du_G/dx), and σ. Equation (OI-1) can therefore be written as:

$$\frac{u_G}{\nu}\frac{d\delta_2^2}{dx}=F\left(\frac{\delta_2^2}{\nu}\frac{du_G}{dx}\right) \qquad \text{(OI-1a)}$$

while (OI-2) becomes

$$\frac{u_G}{\nu}\frac{d\Delta_2^2}{dx}=F\left(\sigma, \frac{\Delta_2^2}{\nu}\frac{du_G}{dx}\right). \qquad \text{(OI-2a)}$$

Here $F(\ldots)$ simply means: "some function of . . ."; of course the function is different in each case.

By analogy, one may suppose that similar equations hold for other boundary layer thicknesses, e.g. δ_1, δ_4, Δ_1, Δ_4. Leaving the subscripts open, these equations may be written as:

$$\frac{u_G}{\nu}\frac{d\delta^2}{dx}=F\left(\frac{\delta^2}{\nu}\frac{du_G}{dx}\right) \qquad \text{(OG-1)}$$

and

$$\frac{u_G}{\nu}\frac{d\Delta^2}{dx}=F\left(\frac{\Delta^2}{\nu}\frac{du_G}{dx}, \sigma\right). \qquad \text{(OG-2)}$$

Two other differential equations which may be considered under the above heading are those introduced by Spalding [7], by way of arguments too devious to be described here. They are:

$$\frac{\sigma}{\gamma}\left(\frac{\delta_4}{u_G}\right)^{1/2}\frac{d}{dx}\left[\Delta_4^3\left(\frac{u_G}{\delta_4}\right)^{3/2}\right]=F\left(\frac{\Delta_4\delta_4}{\nu}\frac{du_G}{dx}\right)$$

and (OG-3)

$$\frac{\sigma}{\nu}\left(\frac{\delta_4}{u_G}\right)^{1/2}\frac{d}{dx}(u_G\Delta_2)^{3/2}=F\left(\frac{\Delta_2^{1/2}\delta_4^{3/2}}{\nu}\frac{du_G}{dx}\right). \qquad \text{(OG-4)}$$

All the equations with OG numbers are exact when $u_G(x)$ obeys the relation mentioned in section 2.2, i.e. when the boundary layers are "similar". Their use in other circumstances involves some inaccuracy.

2.5. *Algebraic equations*

Other equations which are exact for "similar" boundary layers but not in general include those for the thickness ratios. Only the more important ones are listed here. In some cases the ratios are known by special symbols, viz. H_{12}, H_{24}; in others it is necessary to utilize the symbol $H(\ldots)$ without suffix, which signifies: "some function of . . .", as in the case of $F(\ldots)$.

$$\frac{\delta_1}{\delta_2} = H_{12}\left(\frac{\delta_2^2}{\nu}\frac{du_G}{dx}\right) \quad \text{(A-1)}$$

$$\frac{\delta_2}{\delta_4} = H_{24}\left(\frac{\delta_2^2}{\nu}\frac{du_G}{dx}\right) \quad \text{(A-2)}$$

$$\frac{\Delta_4}{\delta_2} = H\left(\frac{\delta_2^2}{\nu}\frac{du_G}{dx}, \sigma\right) \quad \text{(A-3)}$$

$$\frac{\Delta_4}{\Delta} = H\left(\frac{\Delta^2}{\nu}\frac{du_G}{dx}, \sigma\right). \quad \text{(A-4)}$$

These functions, like those appearing in section 2.4, may be tabulated by reference to the solutions of equations (OI-1) and (OI-2). Such tabulations may be found in references [3, 4, 5].

For a "similar" boundary layer it is of course possible to replace du_G/dx in the argument of the OG and A equations by $u_G/[x(1-n)]$, where n, the constant mentioned in section 2.2, characterizes the mainstream flow. Then the "similar" solutions give rise to algebraic equations of the type:

$$\frac{\delta^2}{\nu}\frac{u_G}{x} = F\left(\frac{x}{u_G}\frac{du_G}{dx}\right) \quad \text{(A-5)}$$

and

$$\frac{\Delta^2}{\nu}\frac{u_G}{x} = F\left(\frac{x}{u_G}\frac{du_G}{dx}, \sigma\right). \quad \text{(A-6)}$$

Other algebraic equations have been introduced by particular authors; thus Tifford [8] makes use of:

$$\left(\frac{u_G}{\delta_4}\right)_{\text{effective}} = (0{\cdot}98\sigma^{-0{\cdot}02})^{3/2} \times \left[\left(\frac{u_G}{\delta_4}\right)_{\text{actual}} - \frac{4}{3}\sigma^{-1/4}\frac{\delta_2}{\nu}u_G\frac{du_G}{dx}\right] \quad \text{(A-7)}$$

where of course $u_G/\delta_4 = \tau_S/\mu$ and the suffixes "effective" and "actual" indicate how u_G/δ_4 should be modified to improve the accuracy of Lighthill's method [9] in certain cases. The equation is approximately true for "similar" solutions.

Schuh [10] employs functions connecting β with $(\delta_2^2/\nu)(du_G/dx)$, G with p and $(\delta_2^2/\nu)(du_G/dx)$. All these are obtained from the "similar" solutions.

Merk [11] uses the function

$$\beta = \frac{2}{u_G^2}\frac{du_G}{dx}\int_0^x u_G\, dx. \quad \text{(A-8)}$$

This too is valid for the "similar" solutions: Merk assumes that it is generally valid.

3. METHODS OF PREDICTING HEAT TRANSFER

3.1. *Class number*

We follow Smith and Spalding [12] in classifying calculation methods according to the number of differential equations which are solved: thus Class 1 methods (e.g. Eckert [13]) requires solution of only one equation, while Class 2 methods (e.g. Squire [14]), involve solution of two equations.

The fourth row of Table 1 gives the Class number of each method; which equations are in question may be seen by reference to the entries under "velocity boundary layer", "thermal boundary layer" or both.

In Class 0 falls the method of Stine and Wanlass [15] who predict heat transfer rates without solving any differential equation; these authors make reference directly to equation (A-5) or (A-6) which they assume to hold generally.

3.2. *Exact methods*

The only author whose method of solution is exact is Frössling [16], who solved the partial differential equations (P-1), (P-2) and (P-3) by a series method. Frössling provided general functions permitting the calculation of heat transfer coefficients in particular circumstances as an infinite series. In practice the accuracy of this method is limited by the fact that it is only practicable to use a few terms of the series.

3.3. *Methods using ordinary differential equations*

Most of the methods surveyed in Table 1 require the solution of one or more of the ordinary differential equations listed in sections 2.3 and 2.4. In these equations x is the independent variable, and u_G and du_G/dx are functions of x which are given for a particular problem.

The F-functions appearing in the equations are supposed universal in character; this is why the methods give only approximate results. Usually the F-functions are derived from the "similar" solutions, although other choices may be made; thus linear forms of F-function, approximating those deduced from the "similar" solutions, are very popular because they permit solution of the differential equation as a quadrature.

The solution of the equation itself may proceed by several methods, these include: forward integration, carried out numerically or graphically; a single quadrature which is permissible if the linear F-function is used; and repeated quadrature which gives the accuracy of numerical integration by incorporating an error term into the quadrature procedure.

Often the equation solved yields the distribution of one sort of boundary layer thickness over the body surface, while a different boundary layer thickness is the one of interest; thus $\delta_2(x)$ may have been obtained while $\Delta_4(x)$ is required. In this case the required boundary layer thickness is deduced by subsequent employment of the auxiliary equations represented by (A-1), (A-2), (A-3) and (A-4).

Headings in Table 1 indicate which method uses which practice. For more details, the reader is referred to the original works.

3.4. *Remarks about the methods of Merk* [11] *and Lees* [17]

In the general form of Merk's method [11], the partial differential equations (P-1), (P-2) and (P-3) are solved in a series form; the method is thus an exact one if sufficient terms of the series solution are taken. Unfortunately, few of the coefficients appearing in the solutions have been evaluated; as a result, only the first term in the series can be used. The neglect of the higher terms causes error in the heat-transfer prediction, except when the boundary layer is a "similar" one.

Inspection of Table 1 shows that, in Merk's method, the boundary layer thickness is obtained from the evaluation of a single quadrature followed by the reference to a function derived from the "similar" solutions. The method is thus akin to others which have been classified as belonging to Class 1; this classification is therefore allotted to Merk's method also. Lees' method is put in the same class for the same reason.

4. APPLICATION TO CIRCULAR CYLINDER

4.1. *Description of system*

Each of the methods analysed was applied to the problem of calculating the distribution of the Nusselt number Nu around the leading half of a circular cylinder in a transverse air stream. The following assumptions were made: the flow is laminar; the surface temperature of the cylinder is uniform; the temperature difference between the air stream and the cylinder surface is small so that the fluid properties may be taken as uniform; the Prandtl number for air is 0·7.

The stream velocity distribution outside the boundary layer was taken to be

$$u_G/U = 3{\cdot}631(x/L) - 3{\cdot}275(x/L)^3 - 0{\cdot}168(x/L)^5 \quad (4.1)$$

The coefficients in the above series expansion were given by Eckert [13], who deduced them from measurements of pressure distribution corresponding to a Reynolds number of 170000 by Schmidt and Wenner [1]. Fig. 1 shows the stream velocity distribution represented by equation (4.1).

4.2. *Results of calculation*

In performing the calculations, some numerical integrations were unavoidable; these were carried out using Simpson's Rule, generally with an x/L interval of 0·05.

The results of the computations are presented in graphical form with the heat transfer parameter $Nu/\sqrt{Re}$ as ordinate and the dimensionless distance x/L as abscissa in Figs. 2(a–c). As Frössling's curve can be regarded as an exact one (though its accuracy beyond $x/L = 0{\cdot}45$ is doubtful), it is included in each of the relevant figures to serve as a standard of comparison with other approximate methods.

It will be noted that most calculations were carried out up to $x/L = 0{\cdot}6$ only, due to the limitation of the auxiliary functions available.

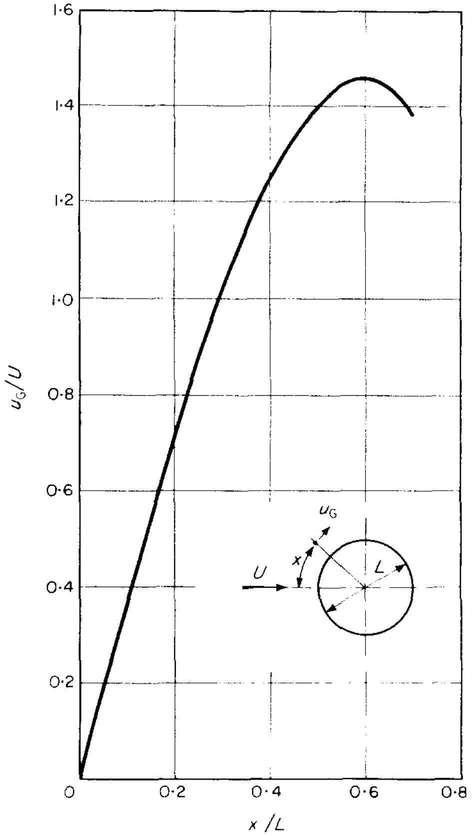

FIG. 1. Velocity distribution outside the boundary layer for a circular cylinder, as represented by equation (4.1).

In any case all methods become of uncertain accuracy close to the separation point, which was estimated to lie between $x/L = 0{\cdot}6$ and $0{\cdot}7$.

4.3. *Experimental results*

For the problem considered in section 4.1 several experimental results are available, e.g. those of Klein [18], Small [19], Kroujilin [20], Schmidt and Wenner [1], Comings *et al.* [21], Giedt [22, 23], Zapp [24], and Kestin *et al* [2].

As has been demonstrated by Comings *et al.* [21], Giedt [23], Zapp [24], and more recently Kestin *et al.* [2], the presence of even small intensities of turbulence in the free stream influences the heat transfer coefficients appreciably. Unfortunately, the experimental data available so far are either for unknown intensities of free stream turbulence or for known intensities of the order of 1 per cent and higher. Thus there are still no experimental data available for situations which are sufficiently close to that postulated in the calculations to act as standards of comparison for the theoretical predictions. As a rough comparison, however, we shall use the experimental data obtained by Schmidt and Wenner [1] for an intensity of free stream turbulence estimated by Kestin *et al.* [2] to be of the order of 0·9 per cent. It was shown by Eckert [13] that the heat transfer coefficients measured by Schmidt and Wenner correlate well into a single curve if those for very high Reynolds numbers are excluded. These average values, taken from [25], are plotted on each of Figs. 2(a–c) along with the various theoretical predictions.

4.4. *Comparison of methods*

Ease of computation. The fifteen methods used can be broadly divided into two groups: group 1 includes methods which require only simple algebra or simple (analytical) integration for the evaluation of local heat transfer coefficients; group 2 includes those which require numerical or graphical integration.

In order of simplicity of use the list of methods under group 1 runs: Frössling, Stine and Wanlass, Lees/Ambrok [26], Skopets [27], Smith and Spalding (for Δ_m), and Merk; that under group 2: Seban/Drake [28/29], Allen and Look [30], Smith and Spalding (for Δ_4), Squire, Eckert/Eckert and Livingood, Schuh, Lighthill, Tifford, and Spalding.

Of course, in the above comparison it is presupposed that for any particular problem the requisite auxiliary functions are available and that the stream velocity is expressible by a power series of finite terms. It seems appropriate to remark here that the methods of Allen and Look, Lees, Ambrok, Squire, Lighthill, Tifford, Schuh, Spalding, Merk (asymptotic series expansion method) and Skopets are not restricted by Prandtl number of the fluid as far as the availability of the auxiliary function is concerned.

ACCURACY

As mentioned earlier the average experimental curve by Schmidt and Wenner cannot be relied

Table 1—Analysis of methods for predicting heat transfer coefficient for laminar uniform-property boundary layer flows

	Name	Frössling	Eckert	Eckert and Livingood	Squire	Allen and Look	Lighthill	Tifford	Seban	Drake	Schuh	Stine and Wanlass	Lees	Ambrok	Smith and Spalding	Spalding	Merk	Skopets
	Reference	16	13	25	14	30	9	8	28	29	10	15	17	26	12	7	11	27
	Date	1940	1942	1952	1942	1943	1950	1951	1950	1953	1953	1954	1956	1957	1958	1958	1959	1959
	Class	2	1		2	1	2	2	1		2	0	1§	1	1	2	1§	2
	Principal assumptions	$u_G(x)$ expressible in a power series of x	Rate of growth of Δ for "non-similar" flows is the same as that for "similar" ones having the same Δ, u_G and (du_G/dx)		Velocity and temperature profiles similar to the velocity profile for flat plate	Reynolds Analogy holds also for σ values slightly different from unity	Linear velocity profile, $u = yu_G/\delta_4$	Modified linear velocity profile	"Non-similar" thermal profiles identical with the "similar" ones corresponding to constant values of $[(\delta_2^2/\nu)du_G/dx]$		Velocity and thermal profiles as for "similar" boundary layers	"Non-similar" velocity and thermal profiles identical with the "similar" ones corresponding to constant values of $[(x/u_G)du_G/dx]$	The zero pressure gradient relation $(d\theta/d\eta)_0 = 0.47\sigma^{\frac{1}{3}}$ where $\eta = yu_G/(2\int_0^x u_G\,dx)^{\frac{1}{2}}$ holds generally	Relation between Δ_4 and Δ_2 depends on σ only and not on pressure gradient	Similar to that for Eckert, Eckert and Livingood	Velocity profile represented by $u = yu_G/\delta_4$ + curvature correction term	"Non-similar" thermal profiles identical with the "similar" ones corresponding to constant values of $(2/u^2)(du_G/dx)\int_0^x u_G\,dx$	Velocity and thermal profiles as for "similar" boundary layers
Velocity boundary layer	Differential equation(s) solved	Partial, P-1 and P-2	—		Ordinary, integral momentum, OI-1	†	Any convenient method used to give u_G/δ_4 a.f.o. x. Then $u = yu_G/\delta_4$ is substituted into the thermal energy equation	As for Lighthill's method	Ordinary integral momentum, OI-1		Ordinary, integral momentum, OI-1	—	—	—	—	Ordinary, integral momentum, OI-1	—	Ordinary, "first moment" integral momentum equation
	Auxiliary functions used in solution	—	—		$(u_G/\nu)(d\delta_2^2/dx)$ a.f.o. $(\delta_2^2/\nu)(du_G/dx)$				$(u_G/\nu)(d\delta_2^2/dx)$ a.f.o. $(\delta_2^2/\nu)(du_G/dx)$		$(u_G/\nu)(d\delta_2^2/dx)$ a.f.o. $(\delta_2^2/\nu)(du_G/dx)$	—	—	—	—	$(u_G/\nu)(d\delta_2^2/dx)$ a.f.o. $(\delta_2^2/\nu)(du_G/dx)$	—	u/u_G a.f.o. y/δ_4
	Source	—	—		4th degree polynomial solutions followed by linear approximation				"Similar" solutions followed by linear approximation		"Similar" solutions followed by linear approximation	—	—	—	—	"Similar" solutions	—	Velocity profile valid for flat plate
	Method of solution	Exact, series	—		Single quadrature				Single quadrature		Single quadrature	—	—	—	—	Repeated quadratures	—	Single quadrature
	Auxiliary functions used subsequently	—	—		$\delta_4/\delta_2 = 2.59$, u/u_G a.f.o. y/δ_4		—	(u_G/δ_4) "corrected" in accordance with equation (A-7)	—		a a.f.o. $(\delta_2^2/\nu)(du_G/dx)$	—	—	—	—	H_{24} a.f.o. $(\delta_2^2/\nu)(du_G/dx)$	—	—
	Source	—	—		Velocity profile valid for flat plate		—	"Similar" solutions (approximated)	—		"Similar" solutions	—	—	—	—	"Similar" solutions	—	—
	Resulting expressions for boundary layer thickness	—	—		$\delta_2^2 = \frac{2.958\nu}{u_G^6}\int_0^x u_G^5\,dx$	$\delta_{D.P.}^2 = \frac{5.3\nu}{u_G^{...}}\int_0^x u_G^{...}\,dx$	—	δ_4 a.f.o. x by any convenient method	$\delta_2^2 = \frac{0.44\nu}{u_G^a}\int_0^x u_G^{a-1}\,dx$ For values of a see notes‡		$\delta_2^2 = \frac{0.441\nu}{u_G^a}\int_0^x u_G^{a-1}\,dx$ $a = 5.165$ for $\frac{\delta_2^2}{\nu}\frac{du_G}{dx} > 0$ $= 5.48$ for $\frac{\delta_2^2}{\nu}\frac{du_G}{dx} < 0$	—	—	—	—	$\delta_2^2 = \frac{0.4418\nu}{u_G^{5.87}}\int_0^x u_G^{4.87}\,dx$ + "error" term $\delta_4 = \delta_2/H_{24}$	—	$\delta_2^2 = \frac{0.45\nu}{u_G^{5.35}}\int_0^x u_G^{4.35}\,dx$
Thermal boundary layer	Differential equation(s) solved	Partial, P-3	Ordinary, OI-2 or OG-2 for Δ_1		Ordinary, integral energy, OI-2	—	Partial, P-3	As for Lighthill's method	—		Ordinary, integral energy, OI-2	—	—	Ordinary, integral energy, OI-2	Ordinary, OG-2 for Δ_4 or Δ_n	Ordinary, equation for growth of Δ, OG-3 or OG-4	—	Ordinary, integral energy OI-2 and "first moment" integral energy equation
	Auxiliary functions used in solution	—	$(u_G/\nu)(d\Delta^2/dx)$ a.f.o.* $(\Delta^2/\nu)(du_G/dx)$ and σ		$\Delta_4/\Delta_1 = \phi(\Delta_1/\delta_4)$, $\Delta_1/\Delta_4 = 0.57$	—	—		—		G a.f.o. p and $(\delta_2^2/\nu)(du_G/dx)$	—	—	$\Delta_4/\Delta_2 = 4.54\sigma^{\frac{1}{3}}$	$(u_G/\nu)(d\Delta^2/dx)$ a.f.o. $(\Delta^2/\nu)(du_G/dx)$ and σ	Velocity profile curvature-correction function $F\frac{\Delta_4\delta_4}{\nu}du_G/dx$	—	$[(u/u_G)$ a.f.o. $(y/\Delta_4)]$ and $[(t/t_G)$ a.f.o. $(y/\Delta_4)]$ a.f.o. $(\delta_2^2/\nu)(du_G/dx)$ and σ
	Source	—	"Similar" solutions		Velocity profile valid for flat plate	—	—		—		"Similar" solutions	—	—	Exact solution for $du_G/dx = 0$	"Similar" solutions	"Similar" solutions	—	"Similar" solutions
	Method of solution	Exact, series	Graphical or numerical		Iterative quadrature	—	Exact, single quadrature		—		Iterative quadrature	—	—	Single quadrature	Repeated quadratures	Repeated quadratures	—	Single quadrature
	Auxiliary functions used subsequently	—	Δ_1/Δ a.f.o. σ and $(\Delta^2/\nu)(du_G/dx)$		$\Delta_1/\Delta_4 = 0.57$	—	...		Δ_4/δ_2 a.f.o. $(\delta_2^2/\nu)(du_G/dx)$ and σ		$\frac{1}{\beta}\frac{\Delta_4\Delta_4}{\nu}\frac{du_G}{dx}$ ... σ	$(\Delta_4^2/\nu)(du_G/dx)$ a.f.o. $(x/u_G)(du_G/dx)$ and σ	—	—	Δ_n/Δ_4 a.f.o. σ and $(\Delta_n^2/\nu)(du_G/dx)$ for Δ_n method only	—	$(\Delta_4^2/\nu)(du_G/dx)$ a.f.o. β and σ	Function $E[(\delta_2^2/\nu)(du_G/dx)]$
	Source	—	"Similar" solutions		Velocity profile valid for flat plate	—		—	"Similar" solutions		"Similar" solutions	"Similar" solutions	—	—	"Similar" solutions	—	"Similar" solutions	"Similar" solutions
	Resulting expressions for boundary layer thickness	$\frac{1}{\Delta_4\sqrt{(UL/\nu)}}\frac{}{L} = \sum_{j=0} A_jK_jX^j$, where $j = 0, 1, 2 \ldots$; A_j are functions of σ only; K_j are constants depending on $u_G(x)$	$\frac{d\Delta_i}{dx} = \frac{\nu}{2u_G\Delta_i}\left(\frac{u_G}{\nu}\frac{d\Delta_i^2}{dx}\right)$ $i = 1$ or 2; and $\Delta_4 = \Delta_i(\Delta_4/\Delta_i)$		$(0.57\Delta_4/\delta_4)^{3}\phi = \frac{0.386\,\nu_G^{\frac{1}{2}}\int_0^x u_G\phi\,dx}{\sigma\phi\int_0^x u_G^{\frac{3}{2}}\,dx}$	$\Delta_4 = \delta_{D.P.}/(\delta_{D.P.}/\delta_4)$ For $du_G/dx \geqslant 0$, $\delta_{D.P.}/\delta_4$ is taken as 0·1765, valid for the flat plate	$\Delta_4 = 1.858\left(\frac{\delta_4}{u_G}\right)^{\frac{1}{3}}\left(\frac{\sigma}{\nu}\right)^{-\frac{1}{3}}\left[\int_0^x\left(\frac{\delta_4}{u_G}\right)^{-\frac{1}{2}}dx\right]^{\frac{1}{3}}$	As for Lighthill's method	$\Delta_4 = \frac{\delta_2}{(\delta_2/\Delta_4)}$		$p = \frac{\left\{\int_0^x G^{\frac{3}{2}}\left[u_G\left(\frac{1}{\beta}\frac{du_G}{dx}\right)^{\frac{1}{2}}\right]^{\frac{3}{2}}dx\right\}^{\frac{2}{3}}}{\left(\frac{2}{3}\sigma\right)^{\frac{1}{3}}\left[u_G\left(\frac{1}{\beta}\frac{du_G}{dx}\right)^{-\frac{1}{2}}G\right]^{\frac{2}{3}}}$ and $\Delta_4 = \frac{p}{G}\left(\nu\beta\Big/\frac{du_G}{dx}\right)^{\frac{1}{2}}$	$\Delta_4 = \left[\nu\left(\frac{\Delta_4^2}{\nu}\frac{du_G}{dx}\right)\Big/\frac{du_G}{dx}\right]^{\frac{1}{2}}$	$\Delta_4 = \frac{\left[\nu\int_0^x u_G\,dx\right]^{\frac{1}{2}}}{0.332\sigma^{\frac{1}{3}}u_G}$	$\Delta_4 = \frac{\left[\nu\int_0^x u_G\,dx\right]^{\frac{1}{2}}}{0.332\sigma^{\frac{1}{3}}u_G}$	For $\sigma = 0.7$ (i) $\Delta_4 = \left[\frac{11.68\nu}{u_G^{2.87}}\int_0^x u_G^{1.87}\,dx\right]^{\frac{1}{2}}$ + "error" term (ii) $\Delta_4 = \frac{\left[\frac{19.78\nu}{u_G^2}\int_0^x u_G\,dx\right]^{\frac{1}{2}}}{\Delta_n/\Delta_4}$	$\Delta_4 = \frac{1.858\nu^{\frac{1}{3}}}{\sigma^{\frac{1}{3}}}\left(\frac{\delta_4}{u_G}\right)^{\frac{1}{3}}\left[\int_0^x\left(\frac{\delta_4}{u_G}\right)^{-\frac{1}{2}}dx \pm \int_0^x\left(\frac{\delta_4}{u_G}\right)^{-\frac{1}{2}}F\,dx\right]^{\frac{1}{3}}$	$\Delta_4 = \left[\frac{(2\nu/u_G^2)\int_0^x u_G\,dx}{\beta\Big/\left(\frac{\Delta_4^2}{\nu}\frac{du_G}{dx}\right)}\right]^{\frac{1}{2}}$ $\beta = \frac{2}{u_G^2}\frac{du_G}{dx}\int_0^x u_G\,dx$	$\Delta_4 = \frac{\left[\nu\int_0^x u_G\,dx\right]^{\frac{1}{2}}}{\sigma^{\frac{1}{3}}E\,u_G}$
	Remarks	Solution exact if sufficient terms used. A few values of A_j for $\sigma = 0.7$ and 1 only are available	—		First approximation often suffices	—	Solution exact if $\Delta_4(\partial^2u/\partial y^2)_0/(\partial u/\partial y)_0$ is close to zero	Solution "almost" exact for flows with $(du_G/dx) \propto u_G^2$	—		—	This procedure was previously mentioned by Eckert and Livingood [25]	The method was originally developed for hypersonic flows for which $T_G/T_S \gg 1$	—	(i) Neglect of "error" term often suffices (ii) This is similar to Merk's formula, though differently derived	This is an improvement of Lighthills' method	—	Solution identical to Lees' and Ambrok's if E is kept constant at the flat plate value

* a.f.o. means "as function of".

† It is not known which equation was solved, the relevant reference being unavailable at time of preparation.

‡ For Seban's $a = 5.12$ for $0 < [(\delta_2^2/\nu)(du_G/dx)] < 0.08$, 5.32 for $-0.2 < [(\delta_2^2/\nu)(du_G/dx)] < 0$;
for Drake's $a = 5.23$ for $0 < [(\delta_2^2/\nu)(du_G/dx)] < 0.09$, 5.32 for $-0.068 < [(\delta_2^2/\nu)(du_G/dx)] < 0$.

§ Also see remarks in section 3.4 of text.

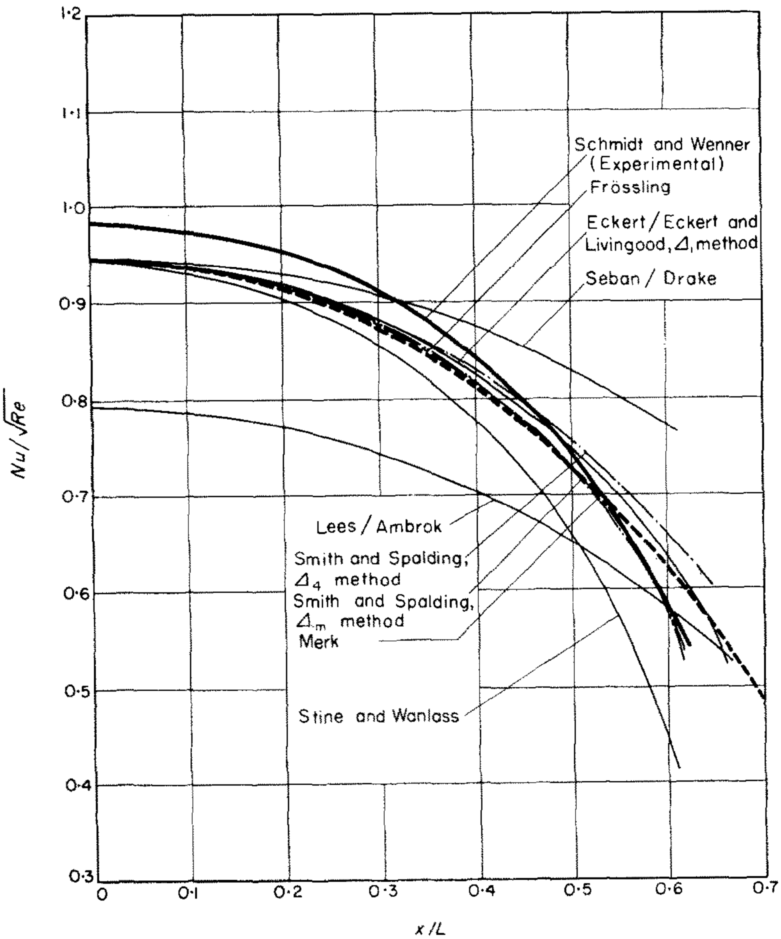

FIG. 2(a). $Nu/\sqrt{Re}$ versus x/L for circular cylinder (σ = 0·7) as predicted by different methods.

upon as a standard of comparison of the various theoretical methods. The discussion following will lean heavily on the fact that up to $x/L = 0{\cdot}45$ Frössling's solution may be regarded as an exact one.

Reference to Fig. 2(a) shows that though the Class 0 method of Stine–Wanlass gives the correct solution at the forward stagnation point, it underestimates the heat transfer coefficients in other regions; the discrepancy is seen to increase with the pressure gradient.

With the exception of Allen–Look's curve, which is shown in Fig. 2(c) in a different scale, all the curves obtained by the Class 1 methods are displayed in Fig. 2(a). It is seen that Allen–Look's method considerably overestimates the heat transfer coefficients; at the forward stagnation point, the predicted value is twice the theoretically correct one. The methods of Lees and Ambrok, which give identical solutions, also have poor accuracy: they underestimate the heat transfer coefficients in the region of negative pressure gradients, but tend to overestimate them in the region of positive pressure gradients. The methods of Eckert/Eckert–Livingood (for Δ_1), Seban/Drake, Smith–Spalding (for both Δ_4 and Δ_m), and Merk all give correct solutions at the forward stagnation point. Except for the curve of Seban/Drake, which clearly has poor accuracy, all the other four curves agree fairly well with that of Frössling throughout the whole range considered. Large discrepancies only

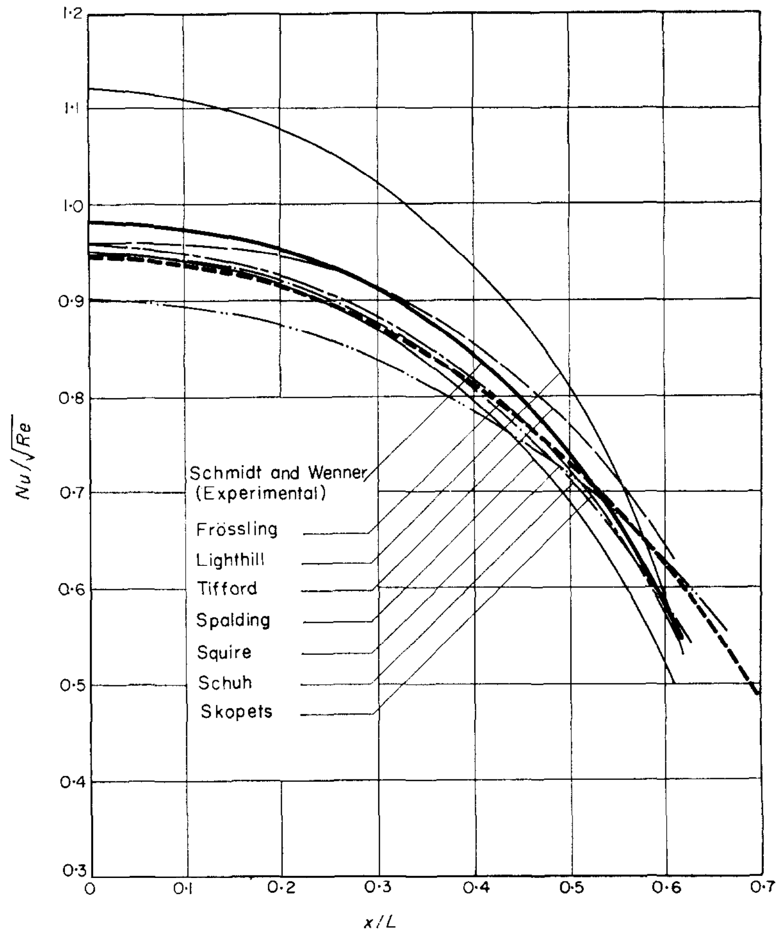

FIG. 2(b). $Nu/\sqrt{Re}$ versus x/L for circular cylinder ($\sigma = 0{\cdot}7$) as predicted by different methods.

occur after $x/L = 0{\cdot}45$, where Frössling's curve can no longer be relied upon as a standard of comparison. If the experimental curve of Schmidt–Wenner is taken as indicating the correct trend for large values of x/L, the methods of Merk and Smith–Spalding (for Δ_m) are the better ones, the difference between these two curves being less than 1 per cent.

Figure 2(b) shows the curves by the Class 2 methods of Frössling, Lighthill, Tifford, Spalding, Squire, Schuh and Skopets. It is seen that Lighthill's method overestimates the heat transfer coefficients, at least in the region of negative pressure gradients, where the discrepancy is seen to increase with decrease of pressure gradient, the maximum being + 18 per cent at the forward stagnation point; the method is quite accurate in the region of small pressure gradients. Tiffords's curve represents an improvement over Lighthill's, but still lies above that of Frössling with a maximum discrepancy of 5 per cent. Spalding's curve agrees well with Frössling's up to $x/L = 0{\cdot}4$, thereafter the two diverge rather rapidly with Spalding's below Frössling's. It is noted that Spalding's curve follows Schmidt–Wenner's experimental one quite faithfully in shape throughout the whole range.

Squire's curve displays the same generally flatter appearance as those of Lees and Ambrok, which also use the flat plate solutions to derive their auxiliary functions. Compared with

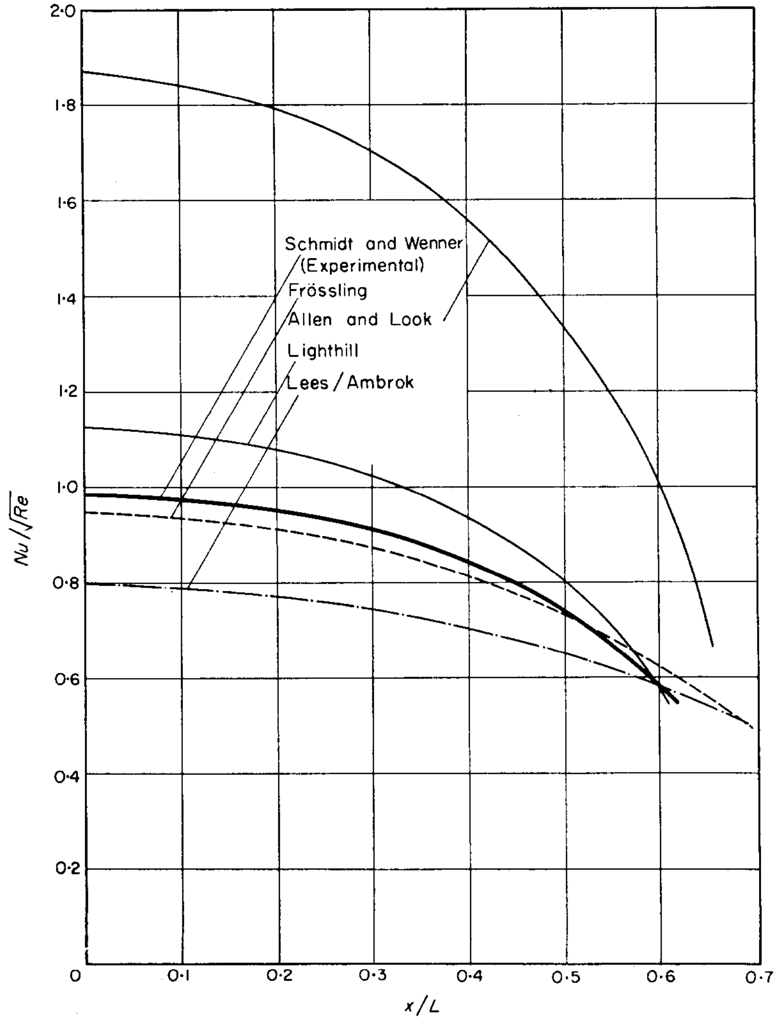

FIG. 2(c). $Nu/\sqrt{Re}$ versus x/L for circular cylinder ($\sigma = 0{\cdot}7$) as predicted by different methods.

Frössling's method, Squire's is quite accurate. The curves of Schuh and Skopets agree fairly well with Frössling's and with those of Smith–Spalding (for Δ_m) and Merk (Fig. 2a).

On a smaller scale, Fig. 2(c) shows the three widely different curves of Allen–Look, Lighthill and Lees/Ambrok together with those of Frössling and Schmidt–Wenner.

To summarize, we list the methods used in order of accuracy based on comparison with Frössling's "exact" solutions in the range $0 \leqslant x/L \leqslant 0{\cdot}5$, as follows:

Method	*Class*	*Accuracy*
Merk	1	within 1–3 per cent
Smith–Spalding (for Δ_m)	1	
Skopets	2	
Schuh	2	
Eckert/Eckert–Livingood (for Δ_1)	1	
Smith–Spalding (for Δ_4)	1	within 3–5 per cent
Squire*	2	
Spalding	2	
Tifford	2	

Stine–Wanlass	0	within 10–20 per cent
Seban/Drake	1	
Lees/Ambrok*	1	
Lighthill*	2	
Allen–Look*	1	within 100 per cent

It should be noted that this table cannot be taken as generally valid; had a surface of different geometry been chosen, the "order of merit" might have been different. However, it is reasonable to suppose that the methods distinguished by an asterisk above will always be relatively poor, since they do not even give the correct answer for the stagnation point. Further, had a problem been considered in which only a part of the periphery was at a different temperature from the mainstream, there is no doubt that the Class 2 methods would prove superior to the Class 1 methods (except perhaps for Skopets' method).

5. CONCLUSIONS

(i) Many methods are available for predicting heat transfer coefficients for laminar, uniform-property boundary layer flows. When applied to a particular problem the accuracy and amount of computational labour involved vary greatly from one method to another. It is, however, not generally true that accuracy increases with computational labour. Often, the choice of the method is dictated by the appropriate auxiliary functions available.

(ii) The methods using "similar" solutions as their auxiliary functions are generally more accurate than those using flat plate solutions alone.

(iii) Accurate experimental data for an intensity of free stream turbulence close to zero are needed before definite conclusions can be drawn as to the relative accuracy of the various methods.

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Résumé—Quinze méthodes de détermination des coefficients de transmission de chaleur en régime laminaire sont étudiées et classées. Chacune de ces méthodes est ensuite appliquée au calcul de la distribution du nombre de Nusselt sur le demi bord d'attaque d'un cylindre circulaire placé dans un écoulement laminaire. Il existe de grandes différences entre les résultats prévus par les théories.

Les résultats sont comparés à la solution exacte de Frössling et aux données expérimentales de Schmidt et Wenner.

Faute de données expérimentales plus sûres on n'a pu tirer aucune conclusions définitives sur la précision relative des méthodes théoriques.

Zusammenfassung—Fünfzehn Methoden zur Berechnung laminarer Wärmeübergangskoeffizienten werden untersucht und klassifiziert. Jede wird zur Berechnung der Verteilung der Nusseltzahl an der Vorderseite eines Kreiszylinders in laminarer Strömung herangezogen. Die Voraussagen der einzelnen Theorien zeigen grosse Unterschiede.

Ein Vergleich wird mit der "exakten" Lösung von Frössling und den Versuchswerten von Schmidt und Wenner aufgestellt. Wegen Fehlens zuverlässigerer Versuchswerte kann die relative Genauigkeit der theoretischen Methoden nicht endgültig beurteilt werden.

Аннотация—Анализируются и классифицируются пятнадцать методов определения коэффициентов теплообмена в ламинарном потоке. Затем каждый из них применяется к задаче расчёта распределения числа Нуссельта на передней половине поверхности круглого цилиндра. Показано, что имеются существенные различия при пользовании разными теориями.

Приводится сравнение «точного» решения Фрёсслинга и экспериментальных данных Шмидта и Веннера. Из-за недостатка более надёжных экспериментальных данных нельзя сделать окончательные выводы об относительной точности теоретических методов.